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Corrigendum

Corrigendum to “An impossibility theorem for wealth in heterogeneous-agent models with limited heterogeneity” [Journal of Economic Theory 182 (2019) 1–24]

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Abstract

This note corrects the proof of Proposition 5 in Stachurski and Toda (2019), which shows that the consumption function has an explicit linear lower bound and is used to prove their main result that wealth inherits the tail behavior of income in Bewley–Huggett–Aiyagari models.

JEL classification: C63; D31; D58; E21

1. Introduction

It has been a ‘folk theorem’ in the quantitative macroeconomics literature that heterogeneous-agent models that feature infinitely-lived agents, constant discount factors, and risk-free asset returns have difficulty in explaining the empirically observed heavy-tailed behavior of the wealth distribution. Stachurski and Toda (2019) (henceforth ST) provide a theoretical explanation by proving that the wealth accumulation process in such models has an AR(1) upper bound (Proposition 6), which implies that the wealth inherits the tail behavior of income (Theorems 3 and 8).

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However, their proof of Proposition 5, which is central to their analysis, contains errors. This note provides a correct proof after slightly strengthening the assumptions.

Although the assumptions used here are stricter than those in ST, they do not exclude the applications that follow the main impossibility theorem (Theorem 8).

2. Assumptions and corrected proof

We consider the following income fluctuation problem:

$$\text{maximize} \quad E_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{1a}$$

$$\text{subject to} \quad a_{t+1} = R(a_t - c_t) + y_{t+1}, \tag{1b}$$

$$0 \le c_t \le a_t, \tag{1c}$$

where $u\colon \mathbb{R}_+ \to \{-\infty\} \cup \mathbb{R}$ is the utility function, $\beta > 0$ is the discount factor, $R > 0$ is the gross risk-free rate, $y_t \ge 0$ is income, a_t is financial wealth at the beginning of period t including current income, and initial wealth $a_0 > 0$ is given.

Following Assumption 1 in ST, the utility function is twice continuously differentiable on $(0, \infty)$ and satisfies $u' > 0$, $u'' < 0$, $u'(0) = \infty$, and $u'(\infty) = 0$. We slightly strengthen Assumption 2 in ST by adding some regularity conditions as in Ma et al. (2020):

Assumption 2'. The income process $\{y_t\}$ takes the form $y_t = y(z_t, \eta_t)$, where $\{z_t\}$ is a Markov chain taking values in a finite set Z with transition probability matrix Π,[1] $\{\eta_t\}$ is IID, y is a nonnegative measurable function, and

$$\sup_{z \in Z} E\left[y(z', \eta') \mid z\right] < \infty \quad \text{and} \quad \sup_{z \in Z} E\left[u'(y(z', \eta')) \mid z\right] < \infty.$$

Note that the transitory shock $\{\eta_t\}$ could be unbounded or heavy-tailed. The condition $\sup_{z \in Z} E\left[u'(y(z', \eta')) \mid z\right] < \infty$, which is not mentioned in ST, is required so that the policy functions remain in the space $\mathcal{C}$ defined below. See Proposition B.4 of Ma et al. (2020) for more details. In all of what follows, Assumption 1 of ST and Assumption 2' above are taken to be in force. We let $S = \mathbb{R}_{++} \times Z$ and take $\mathcal{C}$ to be the set of continuous functions from S to $(0, \infty)$ such that c is increasing in its first argument, $c(a, z) \le a$ for all $(a, z) \in S$, and $\|u' \circ c - u'\| < \infty$, where $\| \cdot \|$ is the supremum norm on S. Given a candidate policy function $c \in \mathcal{C}$, define $(Kc)(a, z)$ to be the unique t in $(0, a]$ that solves the Euler equation

$$u'(t) = \max\{\beta R E\left[u'(c(R(a - t) + y', z')) \mid z\right], u'(a)\}. \tag{2}$$

Under these conditions and $\beta R < 1$, K is a contraction mapping with respect to a complete metric on $\mathcal{C}$. Moreover, its fixed point in $\mathcal{C}$ is the unique optimal consumption policy.[2]

[1] The results presented here can be extended to the case where Z is an abstract metric space under suitable regularity conditions on the stochastic kernel Π, such as those adopted in Li and Stachurski (2014). The case of finite Z is sufficient for the applications we consider.

[2] These facts are proved in Theorem 2.2 of Ma et al. (2020). Although Ma et al. (2020) further assume that Π is irreducible, this assumption is required only for ergodicity and not for optimality. The convergence of a sequence in the metric in question implies pointwise convergence, a fact that we make use of in the proofs below.

Proposition 5 of ST obtains a linear lower bound $c(a, z) \geq ma$ for c with $m > 1 - 1/R$, so that the budget constraint (1b) implies an AR(1) upper bound $a_{t+1} \leq \rho a_t + y_{t+1}$ with $\rho = R(1-m) \in [0, 1)$ for the wealth accumulation process. However, the proof of Proposition 5 in ST contains errors. First, the candidate policy in (A.10) is potentially discontinuous, which is not allowed when applying policy function iteration (in particular, the intermediate value theorem). Second, the argument in Step 3 of the proof requires the inequality to hold pointwise, which is not necessarily true under the stated assumptions.

To correct these errors, we first present a simple lemma, which is essentially a special case of Proposition 2.6 of Ma et al. (2020).

Lemma 1. *If $\beta R < 1$ and there exists an $m \in (0, 1)$ such that*

$$u'(a) \geq \beta R u'(R(1-m)a) \text{ for all } a > 0, \tag{3}$$

then the optimal consumption policy c satisfies $c(a, z) \geq ma$ for all $(a, z) \in S$.

Proof. Let (3) hold for some $m \in (0, 1)$. Define

$$\mathcal{C}_0 := \{c \in \mathcal{C} \mid c(a, z) \geq ma \text{ for all } (a, z) \in S\}.$$

Clearly $\mathcal{C}_0$ is a closed subset of $\mathcal{C}$. Let us show that $K\mathcal{C}_0 \subset \mathcal{C}_0$. To this end, suppose to the contrary that $K\mathcal{C}_0 \not\subset \mathcal{C}_0$. Then by definition there exist $c_0 \in \mathcal{C}_0$ and $(a, z) \in S$ such that $t := Kc_0(a, z) < ma < a$. Since $t < a$, the Euler equation (2) implies

$$u'(t) = \beta R E\left[u'(c_0(R(a-t) + y', z')) \mid z\right].$$

Since $t < ma$, u' is strictly decreasing, $c_0 \in \mathcal{C}_0$, and $y' \geq 0$, we obtain

$$\begin{aligned} u'(ma) < u'(t) &= \beta R E\left[u'(c_0(R(a-t) + y', z')) \mid z\right] \\ &\leq \beta R E\left[u'(m(R(a-t) + y')) \mid z\right] \\ &\leq \beta R E\left[u'(mR(1-m)a) \mid z\right] = \beta R u'(R(1-m)ma). \end{aligned}$$

This contradicts (3) after replacing ma by a. Therefore $K\mathcal{C}_0 \subset \mathcal{C}_0$.

Since clearly the function $c_0(a, z) = a$ is in $\mathcal{C}_0$ and $K : \mathcal{C}_0 \to \mathcal{C}_0$ is a contraction mapping, we have $\mathcal{C}_0 \ni K^n c_0 =: c_n \to c$. Therefore $c \in \mathcal{C}_0$ and $c(a, z) \geq ma$ for all $(a, z) \in S$. $\square$

To apply Lemma 1, we slightly strengthen another assumption in ST. Let

$$\gamma := \sup_{x>0} -\frac{xu''(x)}{u'(x)}. \tag{4}$$

Assumption 3'. The utility function u exhibits bounded relative risk aversion. In particular, γ in (4) is finite.

Proposition 5'. *If Assumption 3' holds and $1 \leq R < 1/\beta$, then the optimal consumption rule satisfies $c(a, z) \geq ma$ for all $(a, z) \in S$, where $m := 1 - \beta^{1/\gamma} R^{1/\gamma - 1} > 1 - 1/R \geq 0$.*

Proof. Since $\beta R < 1$ and $R \geq 1$, we have

$$1 > m = 1 - (\beta R)^{1/\gamma}/R > 1 - 1/R \geq 0.$$

Let $\kappa = \beta R < 1$. The proof of Lemma 11 in ST implies that, if $x > 0$ and $y = (u')^{-1}(\kappa u'(x))$, then $y/x \geq \kappa^{-1/\gamma}$. Since u' is decreasing, we obtain

$$u'(y) = \kappa u'(x) \geq \kappa u'(\kappa^{1/\gamma} y).$$

Setting $y = a$, $\kappa = \beta R < 1$, and noting that $R(1-m) = (\beta R)^{1/\gamma}$, we obtain

$$u'(a) \geq \beta R u'((\beta R)^{1/\gamma} a) = \beta R u'(R(1-m)a).$$

Hence (3) holds. An application of Lemma 1 now yields $c(a, z) \geq ma$. □

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